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ON THE ELECTROMAGNETIC FIELD EQUATIONS IN THE IONOSPHERE

by

HERBERT B. KELLER

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Abstract

This paper consists of two notes on the equations governing the propagation of electromagnetic waves in the ionosphere. In Part I the tensor properties of the ionosphere are derived by a method due to van der Wyck ([1], [4]). While we introduce only minor improvements on his method, it was felt worthwhile to make these results generally more available.

Part II shows the equivalence of various forms of the propagation equations. A general scheme is indicated for finding simplifying transformations of the equations in their first-order from. In particular, we obtain that transformation which yields Rydbeck's coupled equations, and give an interpretation.

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PART I

THE TENSOR PROPERTIES OF THE IONOSPHERE.

1. Introduction

It is well known that the linearized macroscopic field equations in the ionosphere involve tensor properties of the medium. The most accurate derivation of these equations should start with the microscopic field equations, and apply the methods of statistical mechanics to yield the tensor field equations for the macroscopic fields (see[1]). Some such attempts have been made by Hartree [2] and Darwin [3] for a case in which the ionosphere may be described by scalar properties. However, the general derivation in which tensor properties arise seems quite difficult and remains to be carried out. In place of this we are, at present, forced to accept derivations in which the averaging is less detailed. Many such derivations have been given but they are not as rigorous as might be expected, and contain numerous assumptions which are not explicitly stated. In the present note we present a modification of a derivation due to van der Wyck [4] in which the averaging is explicit. The main assumptions are stated and it is shown that the resultant tensor properties of the ionosphere (when some of van der Wyck's simplifying assumptions are eliminated) are just those obtained by the less rigorous methods. This may serve as a justification for the use of the usual derivations. For comparison, one of the standard derivations, essentially that of Banerjea [5], is included.

2. Formulation

The general macroscopic field equations are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{1}$$

together with the relations

$$\vec{B} = \mu \vec{H} \qquad \vec{D} = \epsilon \vec{E} + \vec{P} \qquad \vec{J} = \sigma \vec{E} + \vec{I}. \tag{2}$$

Here μ, ϵ and σ have their usual meanings. $\vec{P}$, the polarization, is a measure of the electrical " stress " of the medium, which is proportional to the average displacement of the electrons from their equilibrium positions. $\vec{J}$, the total current density, consists of the usual conduction current density, $\sigma\vec{E}$, and a convection current density, $\vec{I}$, which is the average flow of free charge density.

The ionosphere is represented by free space, i.e., $\epsilon = \mu = 1$ and $\sigma = 0$, in which there is a given density distribution, $N(x,y,z)$, of free electrons. Heavier particles (ions) are assumed to be present, and we assume their effects may be taken into account by means of the considerations given in Section 4. A static magnetic field, H_E, due to the earth, is also present.

The problem is to derive expressions for the macroscopic quantities $\vec{P}$ and $\vec{I}$ in terms of the field components and ionospheric properties. The equations of motion for charged particles in the presence of electromagnetic fields must play a major part in this derivation; these equations are introduced below.

3. Standard Derivation

In the standard derivation it is assumed that $\vec{I} = 0$ and that

$$\vec{P} = N e \vec{\rho}, \tag{3}$$

where $\vec{\rho}$ is the average displacement of an electron from its equilibrium position and e is the charge on an electron. To compute $\vec{\rho}$ one considers the equation of motion for an "average" electron:

$$\ddot{\vec{\rho}} + \nu \dot{\vec{\rho}} - \frac{e}{m}\left[\dot{\vec{\rho}} \times \vec{H}_E\right] = \frac{e}{m}\vec{E}, \tag{4}$$

where m is the mass of an electron, and where we have used the linearizing approximation $|\vec{H}| \ll |\vec{H}_E|$. The electric force term, $\frac{e}{m}\vec{E}$, is taken here to be the value derived by Darwin [3] and generally used in ionospheric problems, although this has been questioned by some authors. The so-called friction term, $\nu\dot{\vec{\rho}}$, arises from the fact that (4) is the equation of motion for an average electron. It is assumed that collisions of the electrons with the heavier ions occur with average frequency ν. The effect of these collisions is assumed to be a viscous damping, as is indicated by the equation. However, the presence of this term is one of the main points which require a more detailed averaging process. In standard derivations, such as that of the present Section, the friction term is assumed and hence equation (4) is used, while more detailed considerations such as those of Darwin involve the complicated averaging process of statistical mechanics.

If we assume p, E, and H to have time factors* of the form $e^{i\omega t}$, equation (4) becomes

$$(5) \qquad (1 - i\,Z)\,\vec{\rho} + i\,[\vec{\rho} \times \vec{Y}] = -\frac{X}{Ne}\,\vec{E}\ ,$$

where we have introduced the notation

$$(6) \qquad Z \equiv \nu/\omega \ , \quad X = \frac{Ne^2}{m\omega^2}$$

$$\vec{Y} \equiv \frac{e\vec{H}_E}{m\omega} \ , \quad \delta = 1 - i\,Z\ .$$

In matrix form, (5) is

$$(7) \qquad -i \begin{pmatrix} i\delta & -y_3 & y_2 \\ y_3 & i\delta & -y_1 \\ -y_2 & y_1 & i\delta \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \frac{-X}{Ne} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} .$$

Inverting the coefficient matrix yields the average displacement

$$(8) \qquad \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \frac{X}{Ne\,\delta(\delta^2 - Y^2)} \begin{pmatrix} y_1^2 - \delta^2 & y_1y_2 + i\delta y_3 & y_1y_3 - i\delta y_2 \\ y_2y_1 - i\delta y_3 & y_2^2 - \delta^2 & y_2y_3 + i\delta y_1 \\ y_3y_1 + i\delta y_2 & y_3y_2 - i\delta y_1 & y_3^2 - \delta^2 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\equiv \frac{1}{Ne}\,\bar{\bar{T}}\,\vec{E}\ .$$

Here we have introduced the 3×3 matrix $\bar{\bar{T}}$.

The above procedure is invalid when $\delta^2 = Y^2 = 1$; this is a resonance which occurs when the wave frequency ω equals the gyrofrequency $\left| \frac{e\vec{H}_E}{m} \right|$ and when there is no damping.

---

* If the opposite time factor, $e^{-i\omega t}$, had been used, the only effect would be to change the sign of i throughout the remainder of Part 1. The resulting expressions are given in Part II where this time-dependence is assumed.

Substituting (8) in (3) we have

$$\vec{P} = \bar{\bar{T}}\vec{E} \,,$$

and Maxwell's equations (1) become

$$\text{(9)} \qquad \begin{aligned} \nabla \times \vec{E} &= -i\omega \vec{H} \\ \nabla \times \vec{H} &= i\omega \bar{\bar{K}} \vec{E} \end{aligned}$$

where we have introduced the ionospheric dielectric tensor

$$\text{(10)} \qquad \bar{\bar{K}} \equiv \bar{I} + \bar{\bar{T}} \,.$$

Here $\bar{\bar{I}}$ is the unit matrix.

For purposes of comparison with the result of the next Section we may take $y_1 = y_2 = 0$, $y_3 = y$ (i.e., we take $\vec{H}_E$ in the vertical direction), in which case $\bar{\bar{T}}$ becomes

$$\text{(11)} \qquad \bar{\bar{T}} = \frac{X}{(\delta^2 - y^2)} \begin{pmatrix} -\delta & iy & 0 \\ -iy & -\delta & 0 \\ 0 & 0 & -\frac{(\delta^2 - y^2)}{\delta} \end{pmatrix} \,.$$

4. Rigorous Derivation (see: van der Wyck, [4], p. 10-19)

Let $\vec{P} = 0$ and take

$$\text{(12)} \qquad \vec{I} = Ne\,\vec{V} \,,$$

where $\vec{V}$ is the average velocity of electrons due to external electromagnetic field (i.e., a perturbation on the average thermal velocity V_e). To compute $\vec{V}$ we take the coordinates such that $\vec{H}_E = (0,0,H_E)$ and consider the equation of motion for an individual electron between collisions:

$$\text{(13)} \qquad \dot{\vec{v}} - \frac{e}{m}\left[\vec{v} \times \vec{H}_E\right] = \frac{e}{m}\vec{E} \,.$$

Here $\vec{v}$ is the velocity of the electron due to the external fields. We introduce the quantities

$$\text{(14)} \qquad \begin{aligned} v_{\pm} &\equiv v_1 \pm i v_2 \\ E_{\pm} &\equiv E_1 \pm i E_2 \end{aligned}$$

into (13) and obtain, componentwise,

$$(15)\qquad \begin{aligned} \dot{v}_+ + i\omega y\, v_+ &= \frac{e}{m} E_+ \\ \dot{v}_- - i\omega y\, v_- &= \frac{e}{m} E_- \\ \dot{v}_3 &= \frac{e}{m} E_3 \quad . \end{aligned}$$

Here, as in (6), we have used $y \equiv \frac{eH_E}{m\omega}$. From the exponential time dependence of $\vec{E}$, as assumed in Section 3, we must have

$$(16)\qquad E_{\pm} = \hat{E}_{\pm} \sin(\omega t + \phi_{\pm})$$

where $\phi_{\pm}$ are phase constants. Thus the first of equations (15), which is a simple first-order linear inhomogeneous equation, has the general solution

$$(17)\qquad v_+ = e^{-iy\omega t} \left\{ \frac{e}{m} \hat{E}_+ \int e^{iy\omega t} \sin(\omega t + \phi_+)\, dt + \text{const} \right\} .$$

To evaluate the constant it is assumed that $|V_0| \gg |\overline{v}|$, i.e., that the thermal velocity is much larger than the perturbation velocity; hence the velocity $\overline{v}$ after a collision is taken to be zero. If the last collision occurred τ seconds before the time t the solution (17) subject to the above assumption becomes

$$(18)\qquad \begin{aligned} v_+(t,\tau) = \frac{e}{m} \frac{E_+}{(1-y^2)} \Big\{ & \cos(\omega t+\phi_+)\Big[\left(e^{-iy\omega\tau} \cos\omega\tau - 1\right) + iy \left(e^{-iy\omega\tau} \sin\omega\tau\right) \Big] \\ & + \sin(\omega t+\phi_+)\Big[\left(e^{-iy\omega\tau} \sin\omega\tau\right) - iy \left(e^{-iy\omega\tau} \cos\omega\tau - 1\right) \Big] \Big\} . \end{aligned}$$

This solution vanishes when $\tau = 0$, i.e., where a collision has occurred at time t. Now it is assumed that the probability that the electron had a collision in interval $d\tau$ around τ is given by

$$(19)\qquad \frac{V_0}{\ell} \exp\left(-\frac{V_0}{\ell}\tau\right) d\tau \equiv \nu \exp(-\nu\tau)\, d\tau \quad ,$$

where $V_0 \equiv$ average thermal velocity, $\ell \equiv$ mean free path of the electrons, and $\nu \equiv \frac{V_0}{\ell}$ is the average collisional frequency. The average velocity V_+ is then

$$(20) \qquad V_{+}(t) = \int_{0}^{\infty} v_{+}(t, \tau)\, \nu\, e^{-\nu\tau}\, d\tau \quad ,$$

and from (18) we have

$$(21) \quad V_{+} = \frac{e}{m\omega} \hat{E}_{+} \left\{ -\cos(\omega t + \phi_{+}) \left[\frac{1}{1 + (Z+iy)^2} \right] + \sin(\omega t + \phi_{+}) \left[\frac{(Z+iy)}{1+(Z+iy)^2} \right] \right\} .$$

Similarly we obtain

$$(22) \quad V_{-} = \frac{e}{m\omega} \hat{E}_{-} \left\{ -\cos(\omega t + \phi_{-}) \left[\frac{1}{1 + (Z-iy)^2} \right] + \sin(\omega t + \phi_{-}) \left[\frac{(Z-iy)}{1+(Z-iy)^2} \right] \right\} .$$

Now, recalling that $\frac{\partial}{\partial t} E_j = i\omega E_j$, we derive from (14), (21), and (22)

$$(23) \qquad \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{e}{m\omega} \frac{1}{(\delta^2 - y^2)} \left\{ - i\delta \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + y \begin{pmatrix} -E_2 \\ E_1 \end{pmatrix} \right\} .$$

From the last of equations (15) we readily obtain

$$(24) \qquad V_3 = \frac{e}{m\omega} \frac{-i}{\delta} E_3 \; .$$

Equations (23) and (24) substituted in (12) give the current density:

$$(25) \quad \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \frac{i\omega X}{(\delta^2 - y^2)} \begin{pmatrix} -\delta & iy & 0 \\ -iy & -\delta & 0 \\ 0 & 0 & -\frac{(\delta^2 - y^2)}{\delta} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\equiv i\omega \, \overline{\overline{S}} \, \vec{E} \; .$$

Maxwell's equations (1) again reduce to (9) if we introduce

$$(26) \qquad \overline{\overline{K}} \equiv \overline{\overline{I}} + \overline{\overline{S}} \; .$$

From equations (10), (11), (25) and (26) we see that the dielectric tensors agree for the present choice of coordinates. By rotating these coordinates to any arbitrary position it is easily seen that they must agree in general.

The derivation of the present Section is that of van der Wyck [4] up to equations (21) and (22). However, in going to (23) he introduces some assumptions which are apparently unnecessary and make his expressions different from those usually found in the literature.

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PART II

EQUIVALENT FORMS OF MAXWELL'S EQUATIONS FOR THE IONOSPHERE

1. Introduction

Maxwell's equations in the ionosphere have been considered by various authors, and many of these investigations use different forms of the equations for similar configurations of incidence and earth's magnetic field. The reasons for using different but equivalent forms of the equations are varied but generally involve some attempt to reduce the coupling between ordinary and extraordinary waves. In the present note we show a general procedure for obtaining different forms of the field equations. From the method outlined it is easy to see the connection between some well-known forms. In particular the equations for the π - waves of Rydbeck[1] are deduced and the quantities π_1 and π_2 are shown to come from a rotation of the field quantities through a complex angle. A similar treatment for a different ionospheric configuration is considered, and the results seem to be unnecessarily complex. This is an indication that the analysis of Rydbeck is not readily applicable to configurations other than the one he considered.

2. General Formulation

For electromagnetic fields in the ionosphere, with time dependence of the form $e^{-i\omega t}$, Maxwell's equations are

$$\begin{aligned} \nabla \times \vec{E} &= i \frac{\omega}{c} \vec{H} \\ \nabla \times \vec{H} &= -i \frac{\omega}{c} \overline{\overline{K}}(z) \vec{E} \; . \end{aligned} \tag{2.1}$$

The ionosphere is assumed to be horizontally stratified, and

$$\overline{\overline{K(z)}} \equiv (k_{ij}(z)) \equiv \begin{pmatrix} 1-A(\delta^2-y_1^2) & A(y_1y_2+i\delta y_3) & A(y_1y_3-i\delta y_2) \\ A(y_1y_2-i\delta y_3) & 1-A(\delta^2-y_2^2) & A(y_2y_3+i\delta y_1) \\ A(y_1y_3+i\delta y_2) & A(y_2y_3-i\delta y_1) & 1-A(\delta^2-y_3^2) \end{pmatrix} . \tag{2.2}$$

Equations (2.1) and (2.2) are derived in Part I. The notation in equation (2.2) is related to the more standard symbols as follows:

$$\delta \equiv 1 - i\,Z(z) \quad , \quad A = X(z)\,\frac{1}{(1-iZ)\left[(1-iZ)^2-Y^2\right]}$$

(2.3)
$$Z(z) \equiv \frac{\nu(z)}{\omega}$$

$$\vec{Y}(z) \equiv (y_1, y_2, y_3) \equiv \frac{e}{m\omega c}\,\vec{H}_E(z)$$

$$X(z) \equiv \frac{4\pi e^2}{m\omega^2}\,N(z)$$

We seek solutions of equations (2.1) in the form (see [2])

(2.4)
$$\vec{E}(x,y,z) = \vec{V}(z)\,\exp\left(i\,\frac{\omega}{c}\,[px+qy]\right)$$

$$\vec{H}(x,y,z) = \vec{I}(z)\,\exp\left(i\,\frac{\omega}{c}\,[px+qy]\right) \quad .$$

When equations (2.4) are substituted in (2.1) we have

(2.5)
$$\begin{cases} V_3(z) = \dfrac{-1}{k_{33}}\left[pI_2(z)-qI_1(z)+k_{31}V_1(z)+k_{32}V_2(z)\right] \\[2ex] I_3(z) = \left[pV_2(z)-qV_1(z)\right] \end{cases}$$

and

(2.6)
$$\frac{dW(z)}{dz} = i\,\frac{\omega}{c}\,A(z)W(z) \quad .$$

Here we have introduced the matrix notation

(2.7)
$$W(z) \equiv \begin{pmatrix} V_1(z) \\ I_2(z) \\ I_1(z) \\ V_2(z) \end{pmatrix} ; \quad A(z) \equiv \begin{pmatrix} -p\,\dfrac{k_{31}}{k_{33}} & 1-\dfrac{p^2}{k_{33}} & p\,\dfrac{q}{k_{33}} & -q\,\dfrac{k_{32}}{k_{33}} \\[2ex] \left[-q^2+\dfrac{k_{11}k_{33}-k_{13}k_{31}}{k_{33}}\right] & -p\dfrac{k_{13}}{k_{33}} & q\,\dfrac{k_{13}}{k_{33}} & \left[pq+\dfrac{k_{12}k_{33}-k_{13}k_{32}}{k_{33}}\right] \\[2ex] -\left[pq+\dfrac{k_{21}k_{33}-k_{23}k_{31}}{k_{33}}\right] & p\dfrac{k_{23}}{k_{33}} & -q\,\dfrac{k_{23}}{k_{33}} & \left[p^2-\dfrac{k_{22}k_{33}-k_{23}k_{32}}{k_{33}}\right] \\[2ex] -q\,\dfrac{k_{31}}{k_{33}} & p\dfrac{q}{k_{33}} & -1+\dfrac{q^2}{k_{33}} & -p\,\dfrac{k_{32}}{k_{33}} \end{pmatrix}$$

Maxwell's equations for an arbitrarily incident plane wave in the ionosphere have thus been reduced to the first-order system.

However, it is frequently advantageous to introduce new dependent variables which are linear combinations of the V_1, V_2, I_1, and I_2. Thus if $P(z)$ is some non-singular 4×4 matrix we may define the new variable by

$$(2.8) \qquad U(z) = P^{-1}(z)W(z) \quad \text{or} \quad W(z) = P(z)U(z) \quad .$$

The system of equations satisfied by these new variables is, from equations (2.6) and (2.8),

$$(2.9) \qquad \frac{dU(z)}{dz} = \left[i \frac{\omega}{c} P^{-1}AP - P^{-1}\frac{dP}{dz} \right] U(z) \quad .$$

The motivation in choosing a transformation matrix $P(z)$ is usually to put equation (2.9) into a form which can be readily solved (or solved approximately; see [2],[3]). However, in the present note we shall use this method to obtain and compare some equivalent forms of Maxwell's equations for the ionosphere.

3. Normal Incidence (Oblique magnetic field)

For a plane wave normally incident, $p = q = 0$ and the xy - plane may be rotated about the z - axis so that the earth's magnetic field $\vec{H}_E$ shall lie in the yz - plane, say. Therefore, in this case we may take $y_1 = 0$ in eq. (2.3) without loss of generality. The coefficient matrix of the system (2.6) becomes

$$A(z) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & \gamma \\ \gamma & 0 & 0 & -\beta \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad ,$$

where

$$(2.11) \quad \begin{cases} \alpha \equiv 1 - X\delta \dfrac{(\delta - X)}{D} \\ \beta \equiv \left[(\delta^2 - Y^2) - X\delta\right] \dfrac{(\delta - X)}{D} \quad , \qquad D \equiv \left[(\delta - X)(\delta^2 - Y^2) - Xy_2\right] \quad . \\ \gamma \equiv i\, X y_3 \dfrac{(\delta - X)}{D} \end{cases}$$

The term γ serves as a measure of the coupling between "waves" determined by the orthogonal pairs V_1, I_2 and I_1, V_2 of the field components. Thus if γ were zero the system (2.6) would reduce to two independent systems of two equations each. This is readily seen if we eliminate I_1 and I_2 from the system (2.6), using the coefficient matrix A(z) given by (2.10). We then obtain the equations

$$(2.12)\qquad \begin{aligned} \frac{d^2V_1(z)}{dz^2} + \frac{\omega^2}{c^2}\,\alpha(z)\,V_1(z) &= -\frac{\omega^2}{c^2}\gamma(z)\,V_2(z) \\ \frac{d^2V_2(z)}{dz^2} + \frac{\omega^2}{c^2}\,\beta(z)\,V_2(z) &= \frac{\omega^2}{c^2}\gamma(z)\,V_1(z)\ , \end{aligned}$$

which uncouple if $\gamma = 0$. The vertical field components are, from equations (2.5),

$$I_3(z) = 0$$

$$V_3(z) = \frac{-Xy_3}{\left[(\delta-X)(\delta^2-y^2)-\delta y_2^2\right]}\left[i\delta V_1(z) + y_3V_2(z)\right] .$$

These equations have been obtained and studied by Ginsberg [4] and others. The coupling coefficient enters into equations (2.12) in a simple manner, but there is the disadvantage that uncoupling occurs only in the case of a horizontal magnetic field (i.e., when $y_3 = 0$ as in magnetic equatorial regions).

An alternate set of equations can be obtained by the method outlined at the end of Section 2. The motivation for the particular transformation to be used is implicit in the interpretation to be given. Thus we take a variable Lorentz transformation (i.e., rotation through a complex angle) of the electric and the magnetic field components separately. However, since we wish the new components to form two orthogonal pairs, the transformation applied to the magnetic components shall be the inverse of that applied to the electric components. Explicitly we introduce the new variables U(z) as in equations (2.8) with

$$(2.13)\ U(z) \equiv \begin{pmatrix} \pi_1(z) \\ \Psi_1(z) \\ \Psi_2(z) \\ \pi_2(z) \end{pmatrix},\ P(z) \equiv \frac{1}{\sqrt{1-u^2(z)}}\begin{pmatrix} u & 0 & 0 & 1 \\ 0 & -u & 1 & 0 \\ 0 & 1 & -u & 0 \\ 1 & 0 & 0 & u \end{pmatrix},\ P^{-1}(z) \equiv \frac{1}{\sqrt{1-u^2(z)}}\begin{pmatrix} -u & 0 & 0 & 1 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 1 & 0 & 0 & -u \end{pmatrix}$$

The quantity u(z), as yet unspecified, is determined by the condition that $P^{-1}AP$ shall have zero off-diagonal blocks (i.e., that the coefficient matrix in the equation for $U(z)$ be such that the uncoupling of the system is optimal under the given conditions). From the matrix (2.10) and (2.13) we find that this condition requires

$$u^2 + \left[\frac{\alpha - \beta}{\gamma}\right] u + 1 = 0 \quad . \tag{2.14}$$

This equation has two solutions u and $\frac{1}{u}$, where

$$\begin{aligned} u &= \left(\frac{\beta - \alpha}{2\gamma}\right) - \sqrt{\left(\frac{\beta - \alpha}{2\gamma}\right)^2 - 1} \\ &= -i\left\{\frac{y_2^2}{2y_3(\delta - X)} + \sqrt{\left(\frac{y_2^2}{2y_3(\delta - X)}\right)^2 + 1}\right\} \quad . \end{aligned} \tag{2.15}$$

The equation satisfied by $U(z)$ is given by equation (2.9) where, with the above choice of $u(z)$,

$$P^{-1}AP = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\lambda_1^2(z) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^2(z) \\ 0 & 0 & 1 & 0 \end{pmatrix}, -P^{-1}\frac{dP}{dz} = \left[\frac{\dot{u}}{1-u^2}\right]\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{2.16}$$

Here we have introduced the notation $\dot{u} = \frac{du}{dz}$ and*

$$\left.\begin{aligned} \lambda_1^2(z) &\equiv \alpha + \frac{\gamma}{u} \equiv \beta - \gamma u \\ \lambda_2^2(z) &\equiv \alpha + \gamma u \equiv \beta - \frac{\gamma}{u} \end{aligned}\right\} = 1 - \frac{X}{\delta - y_3\left\{\frac{y_2^2}{2y_3(\delta - X)} \pm \sqrt{\left(\frac{y_2^2}{2y_3(\delta - X)}\right)^2 + 1}\right\}} \quad .$$

This system of equations is coupled in a slightly more complicated manner than the previous system but still has only one coupling coefficient. Eliminating Ψ_1 and Ψ_2 from this system yields

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*It should be pointed out that $\pm\lambda_1(z)$ and $\pm\lambda_2(z)$ are the eigenvalues of the matrix $A(z)$ given in equation (2.10). The squares of these eigenvalues are the ordinary and extraordinary dielectric constants, ϵ_1 and ϵ_2, obtained by Rydbeck [1] and others.

$$\frac{d^2\pi_1(z)}{dz^2} + \left[\frac{\omega^2}{c^2}\lambda_1^2(z) + \left(\frac{\dot{u}}{1-u^2}\right)^2\right]\pi_1(z) = -2\left(\frac{\dot{u}}{1-u^2}\right)\frac{d\pi_2}{dz} - \frac{d}{dz}\left(\frac{\dot{u}}{1-u^2}\right)\pi_2$$

$$\frac{d^2\pi_2(z)}{dz^2} + \left[\frac{\omega^2}{c^2}\lambda_2^2(z) + \left(\frac{\dot{u}}{1-u^2}\right)^2\right]\pi_2(z) = -2\left(\frac{\dot{u}}{1-u^2}\right)\frac{d\pi_1}{dz} - \frac{d}{dz}\left(\frac{\dot{u}}{1-u^2}\right)\pi_1 . \tag{2.18}$$

These are Rydbeck's [1] well-known equations and have been examined by various authors. It is clear that the coupling enters in a more complicated manner here than in equations (2.12). However, equations (2.18) uncouple not only in the case of a horizontal magnetic field, but also for a vertical one, since then $y_2 = 0$ and hence $u = -i$. If we start with equations (2.18) then we must use a limiting process for $u \to -i$.

Figure 1 represents the transformation given by equations (2.8) and (2.13).

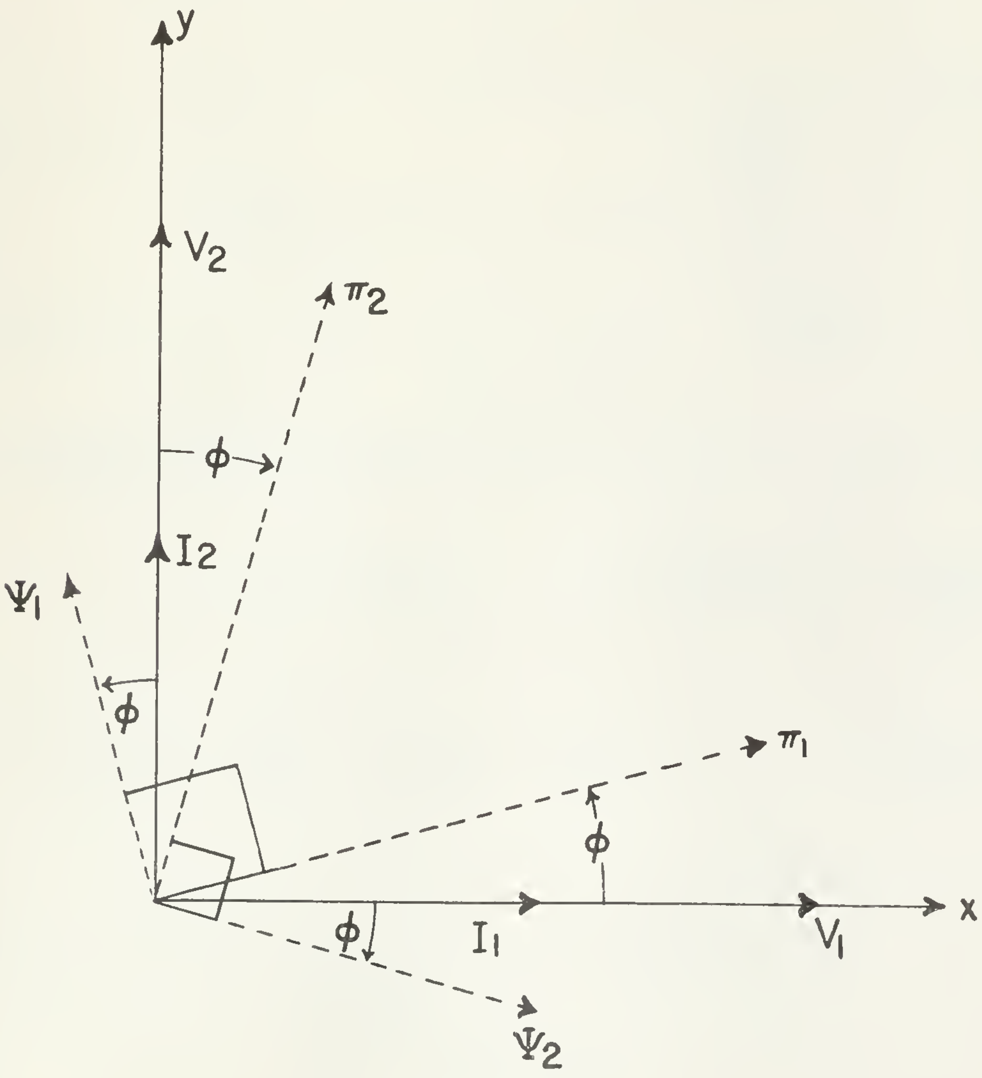

Figure 1

The orthogonal pair of field components V_1, I_2 is essentially replaced by an orthogonal pair π_1, Ψ_1 whose directions are related counterclockwise through an angle ϕ from the directions of the original components. A similar relationship holds between the remaining pairs. However, it should be recalled that the new components are not obtained by a simple rotation of the vectors involved.

4. Oblique Incidence (Vertical magnetic field)

If the wave is normal in the xz - plane we have $q = 0$, and a vertical magnetic field requires $y_1 = y_2 = 0$. The coefficients matrix of the system (2.6) becomes

$$(2.19) \qquad A(z) \equiv \begin{pmatrix} 0 & u & 0 & 0 \\ \alpha & 0 & 0 & \gamma \\ \gamma & 0 & 0 & -\beta \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

where

$$(2.20) \qquad \begin{cases} \alpha = 1 - \dfrac{X\delta}{(\delta^2-Y^2)} \\ \beta = 1 - \dfrac{X\delta}{(\delta^2-Y^2)} - p^2 , \qquad \mu = 1 - \dfrac{p^2\delta}{(\delta-X)} \\ \gamma = i \dfrac{XY}{(\delta^2-Y^2)} \end{cases} .$$

Again we see that if γ were zero there would be no coupling. If we eliminate I_1 and I_2 from the system (2.6) with the above coefficient matrix we have

$$(2.21) \qquad \begin{aligned} \frac{d^2V_1(z)}{dz^2} - \frac{\dot{\mu}}{\mu}\frac{dV_1(z)}{dz} + \frac{\omega^2}{c^2}\mu\alpha V_1(z) &= -\frac{\omega^2}{c^2}\mu\gamma V_2(z) \\ \frac{d^2V_2(z)}{dz^2} + \frac{\omega^2}{c^2}\beta V_2(z) &= \frac{\omega^2}{c^2}\gamma V_1(z) . \end{aligned}$$

The remaining field components are

$$I_3(z) = p V_2(z) ,$$

$$V_3(z) = i \frac{c}{\omega} \frac{p\delta}{\left[(1-p^2)\delta - X\right]} \frac{dV_1(z)}{dz} .$$

The coupling in equations (2.21) is more complicated than that of the system in the previous Section because of the factor μ, which becomes unity for normal incidence. Equations (2.21) are equivalent to those of Wilkes [5] and others. This may be seen by multiplying the first of equations (2.21) by $\frac{1}{\mu}$ and the second by ($\pm$ i) and adding.

The method of Section 2 will be used to obtain an equivalent form of equations (2.21). A comparison of (2.10) with (2.19) suggests a transformation similar to (2.13). However, no choice of u greatly simplifies P^{-1} AP. Thus we modify the transformation by taking U(z) as in (2.13) and

$$(2.22)\quad P(Z) \equiv \frac{1}{\sqrt{1-\mu V^2}}\begin{pmatrix} \mu V & 0 & 0 & 1 \\ 0 & -V & 1 & 0 \\ 0 & 1 & -\mu V & 0 \\ 1 & 0 & 0 & V \end{pmatrix} ; \quad P^{-1}(z) \equiv \frac{1}{\sqrt{1-\mu V^2}}\begin{pmatrix} -V & 0 & 0 & 1 \\ 0 & \mu V & 1 & 0 \\ 0 & 1 & V & 0 \\ 1 & 0 & 0 & -\mu V \end{pmatrix} .$$

We pick V(z) such that P^{-1} AP is simplified as much as possible; this requires (see [2], Section 10) that

$$(2.23)\qquad V^2 + \left(\frac{\mu\alpha - \beta}{\mu\gamma}\right) V + \frac{1}{\mu} = 0 .$$

This equation has the solution V and $\frac{1}{\mu}$ V, where

$$V = \left(\frac{\beta - \mu\alpha}{2\;\gamma}\right) - \sqrt{\left(\frac{\beta - \mu\alpha}{2\mu\gamma}\right)^2 - \frac{1}{\mu}}$$

$$= i\left\{\left(\frac{p^2 y_3}{2\left[\delta(1-p^2)-X\right]}\right) - \sqrt{\left(\frac{p^2 y_3}{2\left[\delta(1-p^2)-X\right]}\right)^2 + \frac{\delta - X}{\delta(1-p^2)-X}}\right\} .$$

Now the coefficient matrix of equation (2.9) has the components

$$(2.25)\qquad P^{-1}\,AP = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^2/\mu \\ 0 & 0 & \mu & 0 \end{pmatrix}$$

and

(2.25) cont'd

$$P^{-1}\dot{P} = \left[\frac{\dot{V}}{1-\mu V^2}\right]\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\mu & 0 \\ 0 & -1 & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix} + \left[\frac{\dot{V}}{1-\mu V^2}\right]\begin{pmatrix} -V/2 & 0 & 0 & 0 \\ 0 & V/2 & -1 & 0 \\ 0 & 0 & -V/2 & 0 \\ 1 & 0 & 0 & V/2 \end{pmatrix}.$$

Here we have introduced

$$\lambda_1^2(z) \equiv \beta - \gamma/V \equiv \mu(\alpha + \gamma V)$$

(2.26)

$$\lambda_2^2(z) \equiv \beta - \gamma\ V \equiv \mu\left(\alpha + \frac{\gamma}{\mu V}\right).$$

Note that again $\pm\lambda_1(z)$ and $\pm\lambda_2(z)$ are the eigenvalues of the $A(z)$ now given by (2.19).

There are now three distinct coupling factors:

$$\frac{\dot{V}}{1-\mu V^2}, \quad \frac{\mu\dot{V}}{1-\mu V^2}, \quad \frac{\dot{\mu}V}{1-\mu V^2}$$

for the system (2.9). They reduce to one if $\mu = 1$ and then we have the case of Section 3. If in eq. (2.9) we use the values given in (2.25) and the first of (2.13) and then eliminate Ψ_1 and Ψ_2, we obtain

$$\frac{d^2\pi_1(z)}{dz^2} - \left[\lambda_1^2 + a^2 + \dot{a} - \frac{c^2}{\mu}\right]\pi_1 = -\left[ab + \dot{b} + \frac{ac}{\mu}\right]\pi_2 - \left[b + \frac{c}{\mu}\right]\dot{\pi}_2\ ;$$

(2.27)
$$\frac{d^2\pi_2(z)}{dz^2} - \frac{\dot{\mu}}{\mu}\frac{d\pi_2(z)}{dz} - \left[\lambda_2^2 + a^2 - \dot{a} - \mu b^2 + \frac{a\mu}{\mu}\right]\pi_2$$

$$= -\left[ac + \dot{c} - \mu ab - \frac{c\dot{\mu}}{\mu}\right]\pi_1 - \left[c + \mu b\right]\dot{\pi}_2 .$$

Here we have introduced

(2.28)
$$a \equiv \frac{\dot{\mu}V^2}{2(1-\mu V^2)}, \quad b \equiv \frac{-\dot{V}}{(1-\mu V^2)}, \quad c \equiv \frac{-(\mu V)^{\bullet}}{(1-\mu V^2)} .$$

The equations (2.27), while equivalent to (2.21), are by no means simpler and seem to have only the virtue that they reduce to the Rydbeck equations (2.18) when $\mu = 1$.

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